Efficiency of nonideal Carnot engines with friction and heat losses

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In nonideal thermodynamic engines the efficiency is well below the Carnot efficiency $\eta = 1 - T_1/T_2$. In 1975 an expression for the efficiency of a nonideal Carnot engine with heat losses was derived, yielding $\eta = 1 - \sqrt{T_1/T_2}$ at maximum power output. In this paper, a corresponding relation is obtained for more general nonideal Carnot engines. If there are friction losses only, the result is $\eta = (1 - T_1/T_2)/2$. If friction and heat losses are both included, the efficiency at maximum power depends on a dimensionless parameter λ^* that takes into account the effects of friction and heat conduction, and can vary between the values obtained for friction and heat losses separately, $(1 - T_1/T_2)/2 < \eta_{\mathrm{pmax}} < 1 - \sqrt{T_1/T_2}$. A general relation between efficiency and power output is established, and it is shown that an appreciable gain in efficiency can be obtained at a power output only slightly below its maximum.
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I. INTRODUCTION

In Carnot engines, a working gas receives heat Q_2 at an upper temperature T_2 from an external heat reservoir at temperature T_2, thereby expanding isothermally from its initial volume V_1 to a larger volume V_2 (step 1). Then the gas is adiabatically expanded to V_3 and cooled to a lower temperature T_1 (step 2). At this temperature it is compressed isothermally to V_4 while rejecting heat Q_1 to an external heat sink at temperature T_1 (step 3). Finally, it is further compressed adiabatically until it reassumes its initial volume V_1 and temperature T_2 (step 4). Because the work that must be expended in the two compression steps (3 and 4) is less than the work gained in the two expansion steps (1 and 2), the net work done during a full cycle is positive and is given by $W = Q_2 - Q_1$. The fraction $\eta = W/Q_2 = 1 - Q_1/Q_2$ of the heat Q_2 that is transformed into work defines the efficiency of the Carnot engine. According to the first and second laws of thermodynamics, the maximum efficiency of an engine that transfers heat from a heat source at temperature T_2 to a heat sink at temperature T_1 is given by

$$\eta_{\mathrm{ideal}} = 1 - T_1/T_2. \tag{1}$$

This value is achieved only by a quasi-static process in which the working gas passes infinitely slowly through a continuous sequence of equilibrium states.

The last condition shows that Carnot engines are idealizations from which all real thermodynamic engines will more or less deviate. In this paper more realistic thermodynamic engines, called nonideal Carnot engines, are considered for which the rate of the various processes is finite, but which otherwise resemble Carnot engines as closely as possible. That is, the working gas is (1) isothermally and (2) adiabatically expanded, and (3) isothermally and (4) adiabatically compressed as in the Carnot cycle. Thereby, the gas passes through almost-equilibrium-states to which a uniform pressure and a uniform temperature can be ascribed in good approximation.

In order for the transformation of heat into work by a Carnot-like engine to proceed at reasonable speed, the heat exchange between the heat reservoirs and the working gas cannot be quasi-static. A temperature gradient is needed, that is, the temperature $\tilde{T}$ of the working gas must be below that of the heat source (T_2) when it receives heat, $\tilde{T}_2 < T_2$, and above that of the heat sink (T_1) when it rejects heat, $\tilde{T}_1 > T_1$. The larger the temperature gradient, the faster the heat is transmitted, the shorter the duration of a full cycle, and the less work W done in one cycle. The average power P of the engine is W/τ, where τ is the period of the cycle. In the range between the maximum work done at infinite period with $P = 0$ and the vanishing work done for the shortest possible τ with $P = 0$, there exists a temperature gradient at which the power output assumes a maximum. For the efficiency at maximum power output, Curzon and Ahlborn derived the temperature dependence[1]

$$\eta_{h,\mathrm{pmax}} = 1 - \sqrt{\frac{T_1}{T_2}}. \tag{2}$$

(The subscript h indicates that only heat losses are taken into account.) This value is appreciably below the efficiency of an ideal Carnot engine given in Eq. (1). Equations (1) and (2) were applied in Ref. 1 to a coal-fired steam plant working between $T_2 = 565$ °C and $T_1 = 25$ °C, yielding $\eta_{\mathrm{ideal}} = 0.64$ and $\eta_{h,\mathrm{pmax}} = 0.40$. The latter is above the observed efficiency of $\eta = 0.36$. In this paper it is shown that the inclusion of friction losses leads to an efficiency that at maximum power output is smaller than the efficiency of Eq. (2), and thus the calculated efficiency may come closer to the efficiencies observed in real engines.

The influence of friction is similar to that of heat conduction. For infinite period the friction losses are negligibly small, but $P = 0$. With increasing speed of the process and simultaneously decreasing efficiency, the friction losses become larger until they eat up the entire work done such that again $P = 0$. In between, there is a process velocity at which the power output becomes maximal while the efficiency is below its maximum.

In Sec. II a formula for the efficiency at maximum power output is derived for the case of friction losses only. In Sec. III losses due to friction and heat conduction will be treated simultaneously.

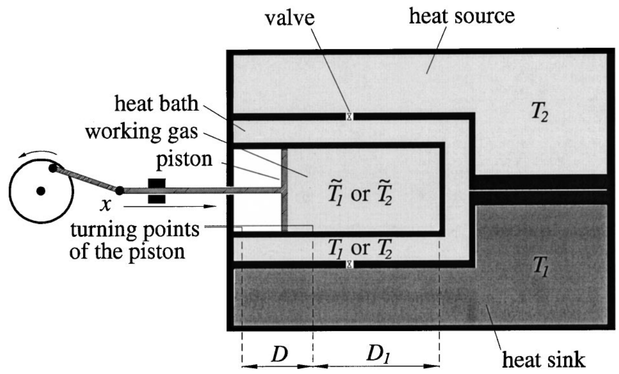

Fig. 1. Model of a nonideal Carnot engine with friction.

II. CARNOT PROCESS WITH FRICTION LOSSES ONLY

A nonideal model engine illustrating the effects to be considered is shown in Fig. 1. In the center there is a hollow cylinder filled with a working gas that is completely closed on its right-hand side. At the left-hand side the rod of a movable piston extends through a wall closing the cylinder to the left. The piston can be pressed into the gas transmitting work to it, and it can be pushed by the gas to the left transmitting work to the outside. The cylinder with the working gas is placed in a larger hollow cylinder that can alternately be filled with the (gaseous or liquid) storage material of a heat sink (temperature T_1) or a heat source (temperature $T_2 > T_1$). It is assumed here that the heat transfer to and from the working gas can proceed at an arbitrary speed without any temperature gradients, so there are no heat losses.

Without friction the efficiency of the engine would be the ideal efficiency of Eq. (1). During the motion of the piston, work is lost due to friction. For a friction force with linear velocity dependence, the total force acting on the piston is

$$F = pA - f\dot{x}, \tag{3}$$

where x is the displacement of the piston. $\dot{x}(t)$ is its velocity, p is the pressure exerted on the piston by the working gas, A is the piston area, and f is the friction coefficient. During a displacement dx, with $v = \dot{x}$ and $dV = A\,dx$, the work performed by the piston is

$$dW = F\,dx = (pA - f\dot{x})\,dx = p\,dV - fv^2(t)\,dt, \tag{4}$$

and the work done during a full cycle is

$$W = \oint p\,dV - f\oint v^2(t)\,dt. \tag{5}$$

We may assume that, for the proper choice of the friction coefficient, $fv^2(t)$ is an averaged rate for all friction losses, including others than the ones caused by friction between the piston and the cylinder wall. According to the first law of thermodynamics, for a complete cycle,

$$0 = \oint dU = \oint \delta Q - \oint p\,dV, \tag{6}$$

and hence

$$\oint p\,dV = \oint \delta Q = Q_2 - Q_1. \tag{7}$$

If we substitute Eq. (7) into Eq. (5), we obtain

$$W = Q_2 - Q_1 - f\oint v^2\,dt. \tag{8}$$

The time dependence of the function $v(t)$ is assumed to be self-similar, that is,

$$v(t) = v^* g(t/\tau), \tag{9}$$

where $g(t/\tau)$ is a periodic function of t/τ with period 1; $g(t/\tau)$ is considered to be a shape function valid for all frequencies $1/\tau$ and characteristic of a given engine, and v^* is a measure of the speed of the process. The distance covered by the piston during one cycle is (see Fig. 1)

$$2D = \oint |v(t)|\,dt = v^*\oint |g(t/\tau)|\,dt = v^*\tau\,\overline{|g(t/\tau)|}, \tag{10}$$

where the bar denotes the time average over one cycle of period τ, i.e.

$$\overline{|g(t/\tau)|} = \frac{1}{\tau}\int_0^\tau |g(t/\tau)|\,dt = \int_0^1 |g(y)|\,dy. \tag{11}$$

Furthermore,

$$\oint v^2(t)\,dt = v^{*2}\oint g^2(t/\tau)\,dt = v^{*2}\tau\,\overline{g^2(t/\tau)}. \tag{12}$$

If we substitute for v^* from Eq. (10), we obtain

$$\oint v^2(t)\,dt = \frac{\overline{g^2(t/\tau)}}{(\overline{|g(t/\tau)|})^2}\,\frac{4D^2}{\tau}. \tag{13}$$

For the Carnot-like engine under consideration, the ratio

$$\delta \equiv \frac{\overline{g^2(t/\tau)}}{(\overline{|g(t/\tau)|})^2} \tag{14}$$

is a fixed parameter. From Eqs. (8), (13), and (14), we obtain

$$W = Q_2 - Q_1 - \frac{4f\delta D^2}{\tau}. \tag{15}$$

According to Eq. (15), the power $P = W/\tau$ of the engine is

$$P = \frac{Q_2 - Q_1}{\tau} - \frac{4f\delta D^2}{\tau^2}, \tag{16}$$

and its efficiency is

$$\eta = \frac{W}{Q_2} = \frac{Q_2 - Q_1}{Q_2} - \frac{4f\delta D^2}{\tau Q_2}. \tag{17}$$

For given values of Q_1 and Q_2 the power P is a function of τ only. Between the shortest possible cycle period with nonnegative P and $\tau = \infty$, P has a maximum, the location of which is determined by

$$\frac{dP(\tau)}{d\tau} = -\frac{Q_2 - Q_1}{\tau^2} + \frac{8f\delta D^2}{\tau^3} = 0, \tag{18}$$

and is given by

$$\tau = \frac{8f\delta D^2}{Q_2 - Q_1}. \tag{19}$$

The maximum power obtained for this period is

$$P_{f,\max}=\frac{(Q_2-Q_1)^2}{16f\,\delta D^2}=\frac{Q_2^2\,\eta_{\text{ideal}}^2}{16f\,\delta D^2}, \tag{20}$$

where $\eta_{\text{ideal}}=1-Q_1/Q_2$ was used. (The subscript f indicates that only friction losses are taken into account.) For the period in Eq. (19) corresponding to maximum power, the use of Eq. (17) with Eq. (1) gives

$$\eta_{f,\text{pmax}}=\frac{1}{2}\left(1-\frac{Q_1}{Q_2}\right)=\frac{\eta_{\text{ideal}}}{2}. \tag{21}$$

Like the Curzon–Ahlborn efficiency of Eq. (2), it is independent of all machine parameters, and, as expected, it is smaller than the ideal efficiency of Eq. (1).

From Eqs. (16) and (17), a general relation between η and P can be derived. For this purpose, we first solve Eq. (16) for τ, yielding

$$\tau=\frac{Q_2-Q_1}{2P}\left(1\pm\sqrt{1-\frac{16f\,\delta D^2 P}{(Q_2-Q_1)^2}}\right). \tag{22}$$

We then use $(Q_2-Q_1)/Q_2=\eta_{\text{ideal}}$ and Eq. (20) to obtain

$$\tau=\frac{Q_2\,\eta_{\text{ideal}}}{2P}\left(1\pm\sqrt{1-\frac{P}{P_{f,\max}}}\right). \tag{23}$$

Because $\eta=W/Q_2=P\tau/Q_2$, it follows that

$$\eta_f=\frac{\eta_{\text{ideal}}}{2}\left(1\pm\sqrt{1-\frac{P}{P_{f,\max}}}\right). \tag{24}$$

The function $\eta_f(P/P_{f,\max})$ has two branches because, according to Eq. (16), every power output $P\neq P_{f,\max}$ can be achieved for a longer and a shorter period, corresponding to a higher and a lower efficiency, respectively. Figure 5 shows this function for the value 0.2 of the ratio

$$\rho=\frac{T_1}{T_2}. \tag{25}$$

III. CARNOT PROCESS WITH COMBINED FRICTION AND HEAT LOSSES

In a nonideal Carnot engine with heat losses only, the working gas does not receive the heat Q_2 at the temperature T_2 of the heat source, but at a temperature somewhat below this value on the isotherm

$$\tilde{T}_2=T_2-\Delta T_2$$

with

$$\Delta T_2>0. \tag{26}$$

The heat Q_1 is not rejected from the gas at the temperature T_1 of the heat sink, but at a temperature somewhat above on the isotherm

$$\tilde{T}_1=T_1+\Delta T_1$$

with

$$\Delta T_1>0. \tag{27}$$

Figure 2 shows a pV diagram of the nonideal process under consideration. On the two isotherms $dU=dQ-p\,dV=0$, and hence

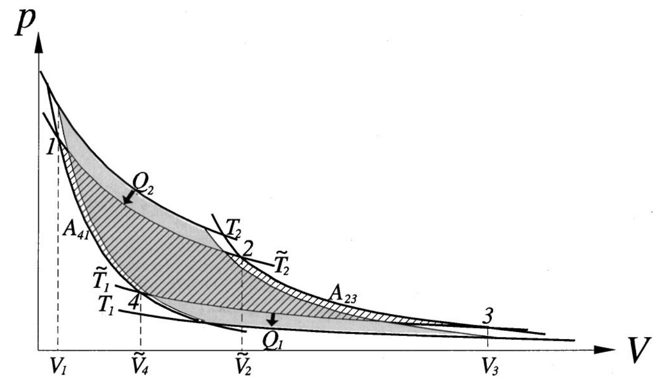

Fig. 2. pV diagram for a nonideal Carnot process with heat losses. The working gas receives the heat Q_2 on an isotherm $\tilde{T}_2=$constant somewhat below the temperature T_2 of the upper reservoir and rejects the heat on an isotherm $\tilde{T}_1=$constant somewhat above the temperature T_1 of the lower reservoir. A_{23} and A_{41} are adiabatic curves. The work W done during one cycle is given by the dashed area in a nonideal process and by the shaded area in an ideal process.

$$Q_1=\int_4^3 p\;dV=R\tilde{T}_1\int_4^3\frac{dV}{V}=R\;(T_1+\Delta T_1)\ln\frac{\tilde{V}_3}{\tilde{V}_4}, \tag{28}$$

and

$$Q_2=R\tilde{T}_2\int_1^2\frac{dV}{V}=R\;(T_2-\Delta T_2)\ln\frac{\tilde{V}_2}{\tilde{V}_1}. \tag{29}$$

It will now be assumed that the Carnot process undergone by the working gas between the temperatures $\tilde{T}_1$ and $\tilde{T}_2$ is an ideal one. From this assumption and the second law of thermodynamics, it follows that

$$\frac{Q_1}{\tilde{T}_1}=\frac{Q_2}{\tilde{T}_2}. \tag{30}$$

Consequently,

$$\frac{\tilde{V}_2}{\tilde{V}_1}=\frac{\tilde{V}_3}{\tilde{V}_4}, \tag{31}$$

according to Eqs. (28) and (29). For the heat transfer between the heat reservoirs and the working gas, Fourier's law of heat conduction leads to $\dot{Q}(t)\sim\Delta T$, where $\dot{Q}$ is the rate of heat transfer. Equivalently,

$$Q_1=\alpha t_1\;\Delta T_1,\quad Q_2=\beta t_2\;\Delta T_2, \tag{32}$$

where t_1 and t_2 are the total times needed for the heat transfer, and where $1/\alpha$ and $1/\beta$ are the thermal resistances between the working gas and the heat sink or the heat source and the working gas, respectively. The engine can be constructed in such a way that t_1 and t_2 are fixed fractions of the period τ. That is, $t_1=c_1\tau$ and $t_2=c_2\tau$, with the consequence that $t_1+t_2=(c_1+c_2)\tau$ or, from Eq. (32),

$$\tau=\xi(t_1+t_2)=\xi\left(\frac{Q_1}{\alpha\Delta T_1}+\frac{Q_2}{\beta\Delta T_2}\right), \tag{33}$$

with $\xi=1/(c_1+c_2)$. In the following, we shall use the abbreviations

$$x=\frac{\Delta T_1}{T_1}, \quad y=\frac{\Delta T_2}{T_2}, \tag{34}$$

$$\mu=\frac{\alpha}{\beta}, \quad P_0=\frac{\alpha T_2}{\xi}, \tag{35}$$

and

$$N=\mu x+y+(1-\mu)xy. \tag{36}$$

If we use Eqs. (26), (27), (30), and (34)–(36), the result for τ in Eq. (33) can be rewritten as

$$\tau=\frac{Q_2}{P_0}\left[\frac{1+x}{x(1-y)}+\frac{\mu}{y}\right]=\frac{Q_2}{P_0}\frac{N}{xy(1-y)}. \tag{37}$$

With this notation the efficiency $\eta_h=(Q_2-Q_1)/Q_1$ of an engine with heat losses only is

$$\eta_h=\frac{Q_2-Q_1}{Q_2}=1-\frac{T_1+\Delta T_1}{T_2-\Delta T_2}=1-\frac{1+x}{1-y}\,\rho, \tag{38}$$

where Eqs. (28), (29), and (31) have been used. We then combine Eqs. (35)–(37) with the relation $P_h=\eta_h Q_2/\tau$ which follows from Eq. (17) and $P_h=W/\tau$, and obtain

$$P_h=\frac{(1-\rho-\rho x-y)xy}{N}\,P_0. \tag{39}$$

If, in addition to heat losses, friction losses are taken into account, Eqs. (16) and (17) must be employed for the power and the efficiency, respectively. The period τ is now determined by heat conduction and is given by Eq. (37). With heat losses included, Eqs. (17), (29), (31), and (35)–(37) lead to the result

$$\eta=1-\frac{1+x}{1-y}\,\rho-\frac{\lambda\ xy}{(1-y)N}, \tag{40}$$

where λ is given by

$$\lambda=\frac{4\alpha f\delta D^2}{\xi R^2 T_2 \ln^2(\tilde{V}_2/\tilde{V}_1)}. \tag{41}$$

Because the power is $\eta Q_2/\tau$, we find using Eqs. (37) and (40) that

$$P=\left[\frac{(1-\rho-\rho x-y)xy}{N}-\frac{\lambda\ x^2y^2}{N^2}\right]P_0. \tag{42}$$

(No subscript will be used for η and P for the case of combined friction and heat losses.)

Except for $\tilde{V}_2/\tilde{V}_1$, all the quantities in λ are fixed machine parameters. For a given engine, the turning points of the piston and its stroke length D are fixed (see Fig. 1). This condition is equivalent to the condition that the minimum volume $\tilde{V}_1$ and the maximum volume $\tilde{V}_3$ have fixed values V_1 and V_3,

$$\tilde{V}_1=V_1, \quad \tilde{V}_3=V_3. \tag{43}$$

On the other hand, the intermediate volumes $\tilde{V}_2$ and $\tilde{V}_4$ depend on the temperatures $\tilde{T}_2$ and $\tilde{T}_1$, respectively. Because $\tilde{V}_2$ is related to $\tilde{V}_3=V_3$ by an adiabatic curve, $TV^{\gamma-1}=\text{constant}$, where γ is the ratio of the specific heats at constant pressure and constant volume, we have $\tilde{T}_2\tilde{V}_2^{\gamma-1}=\tilde{T}_1 V_3^{\gamma-1}$, and

$$\frac{\tilde{V}_2}{\tilde{V}_1}=\frac{V_3}{V_1}\left(\frac{\tilde{T}_1}{\tilde{T}_2}\right)^{1/(\gamma-1)}=\frac{V_3}{V_1}\left(\frac{1+x}{1-y}\,\rho\right)^{1/(\gamma-1)}. \tag{44}$$

Furthermore, the volumes V_1 and V_3 can be expressed in terms of the piston area A and the lengths D_1 and D shown in Fig. 1, that is, $V_1=AD_1$, $V_3=A(D_1+D)$, and

$$\frac{V_3}{V_1}=1+d$$

with

$$d=\frac{D}{D_1}. \tag{45}$$

If we substitute Eq. (45) in Eq. (44) and Eq. (44) in Eq. (41), we finally obtain

$$\lambda=\frac{4\alpha f\delta D^2}{\xi R^2 T_2\left[\ln(1+d)+\frac{1}{\gamma-1}\ln\rho+\frac{1}{\gamma-1}\left(\ln\frac{1+x}{1-y}\right)\right]^2}. \tag{46}$$

To obtain analytical results, we shall use the approximation

$$\lambda\approx\lambda^*\equiv\frac{4\alpha f\delta D^2}{\xi R^2 T_2\left[\ln(1+d)+\frac{1}{\gamma-1}\ln\rho\right]^2}. \tag{47}$$

This approximation is justified for

$$\ln(1+d)+\frac{1}{\gamma-1}\ln\rho\gg\frac{1}{\gamma-1}\ln\frac{1+x}{1-y}, \tag{48}$$

which is satisfied for large values of d and/or γ and/or for small values of x and y. (Large values of γ lead to steep adiabatic curves. It is clear from Fig. 2 that for vertical adiabatic curves, the relation $\lambda\approx\lambda^*$ becomes exact because $\tilde{V}_2=\tilde{V}_3=V_3$ in this case.)

The exact results for η and P are

$$\eta = 1 - \frac{1+x}{1-y}\rho - \frac{\lambda^* xy}{(1-y)N\left[1+\frac{1}{\gamma-1}\left(\ln\frac{1+x}{1-y}\right)\Big/\left(\ln(1+d)+\frac{1}{\gamma-1}\ln\rho\right)\right]^2}, \tag{49}$$

$$P = \left[\frac{(1-\rho-\rho x-y)xy}{N} - \frac{\lambda^* x^2y^2}{N^2\left[1+\frac{1}{\gamma-1}\left(\ln\frac{1+x}{1-y}\right)\Big/\left(\ln(1+d)+\frac{1}{\gamma-1}\ln\rho\right)\right]^2}\right]P_0, \tag{50}$$

where λ^*, given by Eq. (47), is a dimensionless parameter. Instead of these results, we shall use Eqs. (40) and (42), and treat $\lambda \approx \lambda^*$ as a fixed parameter. Note that the two extreme cases, heat losses only ($\lambda = 0$), and friction only ($\lambda \to \infty$), are not affected by our approximation, and remain exact within the framework of the present model.

As mentioned previously, between very slow and very fast cycle speed, there must be a point of maximum power operation. To determine it, the derivatives $\partial P/\partial x$ and $\partial P/\partial y$ must be set equal to zero. Doing so leads to

$$-\rho xyN^2 + (1-\rho-\rho x-y)Ny^2 - 2\lambda xy^3 = 0, \tag{51a}$$

$$-xyN^2 + (1-\rho-\rho x-y)\mu Nx^2 - 2\mu\lambda x^3 y = 0. \tag{51b}$$

We multiply Eq. (51a) by μx^2, Eq. (51a) by y^2, subtract the results, and find $xyN^2(\rho\mu x^2 - y^2) = 0$, or

$$y = \sqrt{\rho\mu}\, x. \tag{52}$$

(For $N=0$, only the uninteresting case $x=0$, $y=0$ would be obtained.) If we substitute Eq. (52) in Eq. (51a), we obtain the cubic equation

$$x^3 + 3\frac{\sqrt{\rho}+\sqrt{\mu}}{\sqrt{\rho}(1-\mu)}x^2 + \frac{-1+2\lambda+3(\rho+\mu)-\rho\mu+4\sqrt{\rho\mu}}{\rho(1-\mu)^2}x = \frac{(1-\rho)(1+\sqrt{\mu/\rho})}{\rho(1-\mu)^2}. \tag{53}$$

We first consider the special case $\mu = 1$ and then arbitrary μ.

Case $\mu = 1$. Because the efficiency $\eta_{h,\mathrm{pmax}}$ for heat losses only does not depend on μ [see Eq. (2)], it might be expected that for the case of combined friction and heat losses, η_{pmax} will depend only weakly on μ. This expectation will turn out to be correct, and hence for many purposes it will be a good approximation to simply set $\mu = 1$.

For $\mu = 1$ the cubic equation, Eq. (53), reduces to a linear one,

$$(-1+2\lambda+3(\rho+1)-\rho+4\sqrt{\rho})x = 2(1+2\sqrt{\rho}+\rho+\lambda) = (1-\rho)(1+\sqrt{1/\rho}), \tag{54}$$

with the solution

$$x = \frac{1-\sqrt{\rho}}{2\sqrt{\rho}[1+\lambda/(1+\sqrt{\rho})^2]}. \tag{55}$$

We substitute this result in Eq. (42) and obtain the maximum power output:

$$P_{\max} = \frac{(1-\sqrt{\rho})^2}{4[1+\lambda/(1+\sqrt{\rho})^2]}P_0. \tag{56}$$

From this result, the maximum power for heat losses only is obtained for $\lambda = 0$, and is given by

$$P_{h,\max} = \tfrac{1}{4}(1-\sqrt{\rho})^2 P_0. \tag{57}$$

For friction losses only, we use Eqs. (20) and (29) with $\Delta T_2 = 0$, $\eta_{\mathrm{ideal}} = 1-\rho$ and Eq. (31) with $\tilde{V}_i = V_i$ to obtain

$$P_{f,\max} = \frac{R^2T_2^2\ln^2(V_2/V_1)}{16 f\delta D^2}. \tag{58}$$

If we use $V_2 = V_3(t_1/T_2)^{1/(\gamma-1)}$ for adiabatic processes, the second equation of Eq. (35), Eq. (45), and Eq. (47) $P_{f,\max}$ becomes

$$P_{f,\max} = \frac{(1-\rho)^2 P_0}{4\lambda}. \tag{59}$$

Figure 3 shows the dependence of $P_{\max}$ and $P_{f,\max}$ on λ for fixed ρ; $P_{h,\max}$ also is shown for comparison. For large values of λ, $P_{\max}$ and $P_{f,\max}$ agree approximately. For $\lambda \to 0$, however, $P_{\max}$ does not converge to $P_{f,\max}$. The reason is that in our model of Carnot-like engines with friction losses only, we assumed that the cycle can be passed through at arbitrary speed without any temperature gradients; consequently $P \to \infty$ for $f \to 0$ or $\lambda \to 0$, respectively, while for combined heat and friction losses P is kept finite by the finite period needed for heat conduction.

From Eq. (40) with Eq. (52) for $\mu = 1$,

$$\eta = \frac{1-\sqrt{\rho}-\sqrt{\rho}[1+\lambda/(1+\sqrt{\rho})^2]x}{1-\sqrt{\rho}x}(1+\sqrt{\rho}) \tag{60}$$

is obtained. If we substitute Eq. (55), we find

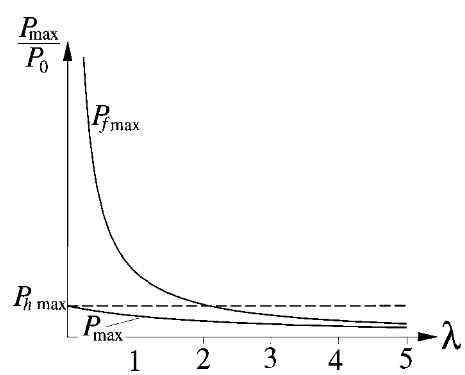

Fig. 3. $P_{\max}/P_0$ as a function of λ for friction losses only (curve $P_{f,\max}$) and the case of combined friction and heat losses (curve $P_{\max}$) with $\rho = 0.2$. The value $P_{h,\max}$, which is independent of λ, is shown for comparison (dashed line). (The power P_0 defined in the second equation of Eq. (35) is a reference value that is typical for a specific engine.)

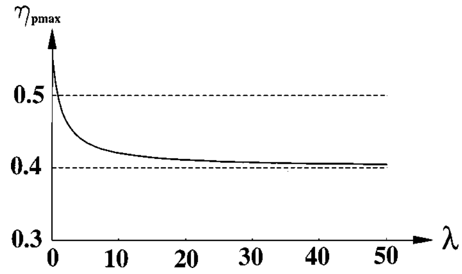

Fig. 4. Efficiency $\eta_{\rm pmax}$ at maximum power output as a function of λ for $\rho=0.2$. The highest efficiency, $\eta_{\rm pmax}=0.55$, is obtained for heat losses only $(\lambda=0)$, the smallest, $\eta_{\rm pmax}=0.4$, for friction losses only.

$$\eta_{\rm pmax}=\frac{1+\lambda/(1+\sqrt{\rho}\,)^2}{1+2\lambda/(1+\sqrt{\rho}\,)^3}\,(1-\sqrt{\rho}\,). \tag{61}$$

For vanishing friction $(\lambda=0)$, Eq. (61) reduces to Eq. (2), that is $\eta_{\rm pmax}=\eta_{f,\rm pmax}$, in contrast to the behavior of $P_{\rm pmax}$. (The reason is that τ does not enter the thermal contribution to η but enters that of P.) If friction dominates (large values of λ), $\eta_{\rm pmax}$ approaches Eq. (21). For intermediate values of λ, Eq. (61) interpolates between friction losses only and heat losses only. Figure 4 shows the dependence of $\eta_{\rm pmax}$ on the friction parameter λ for the given temperature ratio $\rho=0.2$. It is seen that at maximum power output, the highest efficiency is obtained for $\lambda=0$ (heat losses only), and the lowest efficiency is obtained for $\lambda\to\infty$ (friction losses only). In all real engines a situation between these two extreme cases will be found.

For $\mu=1$ the general relation between efficiency and power given by Eq. (24) for friction losses only can easily be generalized to the case of combined friction and heat losses. However, a problem is posed by the fact that both P and η depend on two variables, x and y, and hence a unique relation between P and η does not exist. Nevertheless, such a relation can be established for every curve $y=y(x)$ in the space of the variables x and y. In the following a relation of this kind will be derived for the special curve $y=\sqrt{\rho}\,x$ that results from Eq. (52) for $\mu=1$ and runs through the point of maximum power output in the x,y plane.

For $\mu=1$ and $y=\sqrt{\rho}\,x$, Eq. (42) leads to

$$\frac{P}{P_0}=\sqrt{\rho}\,(1-\sqrt{\rho}\,)\,x-\rho\,[1+\lambda/(1+\sqrt{\rho}\,)^2]\,x^2. \tag{62}$$

Equation (62) is a quadratic equation in x with the solution

$$x=\frac{1-\sqrt{\rho}}{2\sqrt{\rho}\,[1+\lambda/(1+\sqrt{\rho}\,)^2]}\times\left(1\pm\sqrt{1-\frac{4(P/P_0)\,[1+\lambda/(1+\sqrt{\rho}\,)^2]}{(1-\sqrt{\rho}\,)^2}}\right). \tag{63}$$

With the help of Eq. (56), the latter can be transformed to

$$x=\frac{1-\sqrt{\rho}}{2\sqrt{\rho}\,[1+\lambda/(1+\sqrt{\rho}\,)^2]}\,(1\pm\sqrt{1-P/P_{\rm max}}\,). \tag{64}$$

If this result is inserted in Eq. (60), we obtain after simple algebraic manipulations:

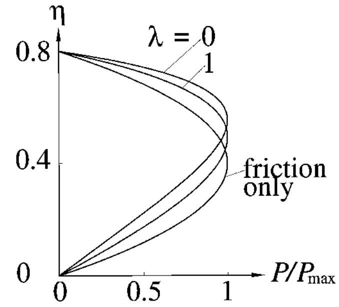

Fig. 5. Efficiency η as a function of the power ratio $P/P_{\rm max}$ for the case of friction losses only, of heat losses only $(\lambda=0)$, and for the case of combined friction and heat losses with $\lambda=1$. The parameter values are $\rho=T_1/T_2=0.2$ and $\mu=1$.

$$\eta=\frac{[1+\lambda/(1+\sqrt{\rho}\,)^2](1-\sqrt{\rho}\,)(1\pm\sqrt{1-P/P_{\rm max}}\,)}{1\pm\sqrt{1-P/P_{\rm max}}\,(1-\sqrt{\rho}\,)/(1+\sqrt{\rho}\,)+2\lambda/(1+\sqrt{\rho}\,)^3}. \tag{65}$$

Figure 5 shows η for a fixed value of $\rho=T_1/T_2$ as a function of $P/P_{\rm max}$ for several values of λ. Included are the special cases of friction and heat losses only. The first, given by Eq. (24), is obtained from Eq. (65) for large λ. The second is obtained for $\lambda=0$ and has the efficiency

$$\eta_h=\frac{(1-\sqrt{\rho}\,)(1\pm\sqrt{1-P/P_{h,\rm max}}\,)}{1\pm\sqrt{1-P/P_{h,\rm max}}\,(1-\sqrt{\rho}\,)/(1+\sqrt{\rho}\,)}. \tag{66}$$

Because according to Eq. (65) η varies only slowly with $P/P_{\rm max}$ near $P=P_{\rm max}$, an appreciable increase of η can be achieved at a power only slightly below its maximum. (For heat losses only, this result was shown somewhat more generally in Ref. 2. See also Refs. 3 and 4.)

Arbitrary μ. In this case the cubic equation, Eq. (53), must be solved. Because its right-hand side is positive while its left-hand side vanishes for $x=0$ and goes to ∞ for $x\to\infty$, there always exists one positive solution. Because the quadratic term is positive, the other two solutions are either negative or complex, and because $x=\Delta T_1/T_1>0$, the positive solution must be chosen. For $\lambda\leqslant(1+\sqrt{\rho\mu})^2/2$, the solution is found to be

$$x=\frac{1}{\sqrt{\rho}\,(1-\mu)}\left[\frac{2}{\sqrt{3}}\,(1+\sqrt{\rho\mu})\times\sqrt{1-\frac{2\lambda}{(1+\sqrt{\rho\mu})^2}}\cos\frac{\varphi}{3}-(\sqrt{\rho}+\sqrt{\mu})\right], \tag{67}$$

with

$$\varphi=\arccos\frac{3\sqrt{3}\,(\sqrt{\rho}+\sqrt{\mu})\,\lambda}{(1+\sqrt{\rho\mu})^3[1-2\lambda/(1+\sqrt{\rho\mu})^2]^{3/2}}. \tag{68}$$

To obtain $\eta_{\rm pmax}$, this result must be substituted in the equation resulting from Eq. (40) with y expressed in terms of x using Eq. (52):

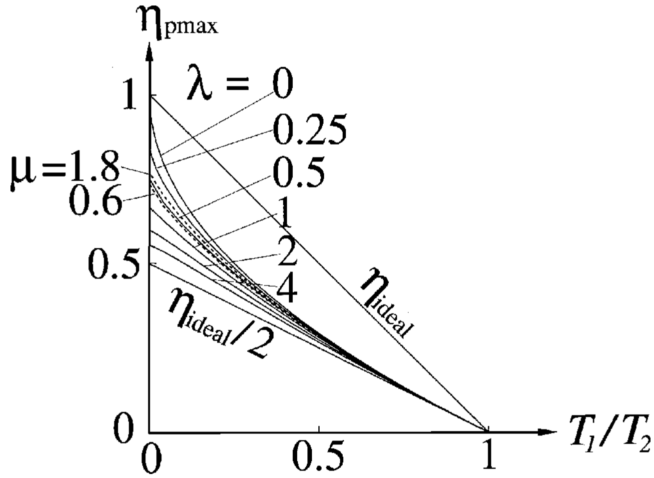

Fig. 6. Carnot efficiency $\eta_{\rm ideal}$ and efficiency η at maximum power output as a function of the temperature ratio $\rho=T_1/T_2$ for the case of heat losses only $(\lambda=0)$, friction losses only (lowest curve), and for the case of combined friction and heat losses. The solid curves are for $\mu=1$ and the dotted curves for $\mu\neq 1$, $\lambda=0.5$. It can be seen that the value of μ does not have much influence on the results.

$$\eta=1-\frac{1+x}{1-\sqrt{\rho\mu}\,x}\,\rho-\frac{\lambda\,\sqrt{\rho}\,x}{(1-\sqrt{\rho\mu}\,x)[\sqrt{\rho}+\sqrt{\mu}+\sqrt{\rho}\,(1-\mu)\,x]}. \tag{69}$$

Figure 6 shows the temperature dependence of the efficiency at maximum power obtained from Eq. (69) with Eq. (67) for $\lambda=0.5$, $\mu=0.6$, and $\mu=1.8$. As suggested earlier, $\eta_{\rm pmax}$ depends only weakly on μ.

Because the curves in Fig. 6 corresponding to $0<\lambda<\infty$ must be squeezed between the curves for $\lambda=0$ and $\lambda=\infty$, and because the latter ones are exact within the framework of the present model, they would not be much different even if the conditions for the validity of the approximation of Eq. (47) were not satisfied very well.

IV. SUMMARY AND CONCLUSIONS

The efficiency of nonideal Carnot engines was determined by taking into account the effects of friction and heat conduction. In the regime between maximum efficiency at zero process speed, and zero efficiency at the largest possible speed without net energy losses, there exists a speed at which the power output of the engine becomes maximal. For the case of heat losses only, this result was shown previously in Ref. 1; the corresponding efficiency was found to be $\eta_{h,\rm pmax}=1-\sqrt{T_1/T_2}$. When only friction losses are taken into account, $\eta_{f,\rm pmax}=(1-T_1/T_2)/2$ was found in this paper. When both effects are taken into account, the efficiency at maximum power output depends on a dimensionless parameter λ^* that includes the effects of friction and heat conduction. An approximation was introduced to obtain analytical results for intermediate values of λ^*. In addition, an exact treatment in the framework of the model was discussed.

The efficiency can vary between a lower limit, $\eta_{f,\rm pmax}=(1-T_1/T_2)/2$, obtained for friction losses only, and an upper limit, $\eta_{h,\rm pmax}=1-\sqrt{T_1/T_2}$, for heat losses only. Within the framework of the model these two limits are exact. For a temperature drop from 565 to 25 °C, these limits mean that $0.32<\eta_{\rm pmax}<0.40$. That is, the efficiency obtained for heat losses only can be lowered by friction by up to 20% in this case. In reality, friction and heat losses will always both be present, and hence $\eta_{\rm pmax}$ will assume a value in between the two extreme cases. We conclude that to obtain the highest possible efficiency, friction losses should be reduced as much as possible.

The parameter λ^* depends on rather general engine properties, so it may turn out to be useful for the characterization of other thermodynamic engines as well.

In addition to the determination of $\eta_{\rm pmax}$, a general relation between η and P was derived. From this relation, it can be seen that only a small reduction of P below $P_{\max}$ can lead to an appreciable increase of the efficiency. For example, for $\rho=0.2$ and $\lambda=1$, the efficiency is increased by 16% if the power is lowered by 5%.

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